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Programming Under Uncertainty and Stochastic Optimal Control

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PROGRAMMING UNDER UNCERTAINTY

AND

STOCHASTIC OPTIMAL CONTROL

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ABSTRACT

This paper extends the theory of Programming under Uncertainty to the case when the decision variables are elements of a Banach space. This approach leads to a very natural application of the computational techniques of mathematical programming to stochastic optimal control problems. It is shown that there exists an equivalent deterministic mathematical program whose set of feasible solutions is a convex set and whose objective function can be expressed as a convex function of the initial decision variables. In the second part, a duality theory is developed for this class of problems and some of the relations to the maximum principle for stochastic linear control problems are given.

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1. INTRODUCTION

Most optimization models (programming models, optimal control models, etc.) assume that the model's parameters (coefficients, functions, etc.) are well specified, either as best estimates, or by their expected values, and so on. In reality, however, these quantities are subject to uncertain or random variations of various kinds due to noise, component failure, unexpected demands, etc. Such discrepancies between reality and model can be reduced by assuming that all or some of the parameters are random variables with known probability distribution function.

Unfortunately, the complexity of such models, and of their solution, increases rapidly with the "amount" of uncertainty present in the problem. Nonetheless, different approaches and different techniques have given us some grip on a certain class of problems, for which there exist now "efficient" solution methods.

A. Programming under Uncertainty

In 1955, G. Dantzig formulated the two-stage linear program under uncertainty model [2]. The theory was furthered by G. Dantzig and A. Madansky [3], A. Madansky [5], R. Wets [8], and some special cases were investigated by A. Williams [9], [10] and R. Wets [7].

The *standard form* of a programming under uncertainty problem reads,

$$\begin{array}{llcccl}
\text{Minimize} & z(x) = cx + E_\xi\{\text{Min } qy\} & & & & \\
\text{subject to} & Ax & & = b & & \\
 & Tx & + & My = \xi & & \text{on } (\Xi, \mathcal{F}, F) \\
 & x \geq 0, & & y \geq 0. & &
\end{array}$$

where A, T, and M are fixed matrices, c, q, b are constant vectors, x and y are variables, and ξ is a random vector defined on the probability space $(\Xi, \mathcal{F}, F)$. The only random parameter present in this problem is ξ. The decision process described by this model is a two-stage process in which one first selects x, then observes ξ and finally selects y so as to satisfy the constraints of the problem. The decision process is thus divided into two parts, but only the first one is of interest since once x is selected and ξ is observed, finding Inf of qy subject to $My = Tx - \xi$, $y \geq 0$ is a deterministic problem. One procedure to solve such a problem is to exhibit a deterministic problem, whose set of optimal solutions is identical to the set of optimal solution of our original problem. In general, such a deterministic problem exists, and it is shown that it has the form of a convex program. To find an explicit expression for this *equivalent* convex program is not always trivial, but it is possible to do so for an important class of problems [7].

B. Sequential Decision Processes and Stochastic Optimization

It is not difficult to see that the two-stage programming under uncertainty problem can be generalized to a n-stage decision process where we have a sequence of decisions, observance of the behavior of the system and new decisions (corrective action). This idea is not new but literally illustrated by Dynamic Programming. Many stochastic optimization problems fall naturally in this framework, even if sometimes

the concept of decision stage may only be a mathematical fiction, see [8, II. A].

C. <u>The Stochastic Optimal Control Problem</u>

Usually, the stochastic optimal control problem is also formulated in the framework of a sequential decision process. But rather than dealing with a finite number of stages, it is assumed that the corrective actions are taken at every instant, i.e. at an infinite number of stages. To see this, it suffices to remark that a solution (control) for such a problem is not only expressed as a function of time, but also as a function of the actual state of the system [1], [4]. The observed state of the system consists then of the space-state determined by the control function affected by the interference of a random (noise) process.

In order to obtain an explicit expression for the solution of such a problem, or find an algorithmic procedure leading to the solution, different assumptions have been made, explicitly or implicitly in the formulation of the problem. From a practical point of view, probably one of the weakest assumptions one could make is to assume that the number of corrective actions is finite, either at fixed time intervals or at some time intervals to be determined by the control system itself.

An n-stage control system can be described as follows: Let $x(t)$ describe the space state obtained by controlling the system with $u_1(t)$ for $0 \leq t \leq t_1$. Let $y(t)$ be the observed state of the process, i.e.

$y(t) = x(t) + \xi(t)$ where $\xi(t)$ is a random (noise) function. If $u_2(t)$ is the second stage control for $t_1 \leq t \leq t_2$, we have $u_2(t) = \phi(t, y_1(t))$ or $\hat{\phi}(t, u_1(t), \xi(t_1))$ and similarly for $t_2 \leq t \leq t_3$, we have $u_3(t) = \psi(t, y(t_2))$ or $\hat{\psi}(t, u_1(t), \xi(t_1), u_2(t), \xi(t_2))$, and so on.

This structure is underlying our approach to the stochastic optimal control problem. We develop the theory for a two-stage system but the generalization to an n-stage process presents no mathematical difficulty. In Section II, we derive the deterministic equivalent of the stochastic problem. Section III is devoted to a duality theory for this class of problems and its relation to the maximum principle. A projected paper will deal with the applications of the theoretical results obtained here to specific control problems.

2. THE EQUIVALENT CONVEX PROGRAM

A. The Problem

The *standard form* of the problem to be considered in this paper is:

$$\text{(1)}\quad \text{Find} \quad \inf z(u) = c(u) + E_\xi\{\text{Inf } q(v[\xi])\}$$
$$\text{subject to} \quad A(u) = b$$
$$T(u,\xi) + W(v[\xi]) = d$$

where

- u is restricted to lie in some closed convex subset U of a Banach space $\mathcal{U}$ and, $v[\xi]$ must belong to a closed convex subset V of a Banach space $\mathcal{V}$ for each ξ.
- b and d are points in $\mathfrak{R}^m$ and $\mathfrak{R}^{\bar{m}}$, respectively.
- ξ is a random variable defined on a probability space $(\Xi,\mathcal{F},F)$, note that $v: \Xi \to \mathcal{V}$.
- c and q are continuous convex functionals on $\mathcal{U}$ and $\mathcal{V}$, respectively.
- A,T,W are continuous linear operators such that

$$A : \mathcal{U} \to \mathfrak{R}^m$$
$$T : \mathcal{U} \times \Xi \to \mathfrak{R}^{\bar{m}}$$
$$W : \mathcal{V} \to \mathfrak{R}^{\bar{m}}$$

- The operator E_ξ stands for expectation of the inf of $q(v[\xi])$ with respect to ξ.

The process described by Problem (1) can be interpreted as follows: We first select a point in U, satisfying the constraints $A(u) = 0$ and $u \in U$, say $\hat{u}$; we then observe the random event, say $\hat{\xi}$, and we are finally allowed to pick a point of V such that $v \in V$, $W(v) + T(\hat{u},\hat{\xi}) = 0$ and $q(v)$ is minimum. The decision process is thus divided into two stages. The second-stage decision is taken, when no uncertainties are left in the problem, i.e. when the random variable has been observed. This second stage is not our immediate interest here. Our primary interest is to find a *"feasible"* u which minimizes our total cost. Not only does our objective function take into account the immediate cost: $c(u)$, but also a weighted average of the cost of all the optimal second-stage decision, a given u, may lead to.

For the sake of simplicity, we shall assume that $(\Xi, \mathcal{F}, F)$ is the probability space induced in $\mathfrak{R}^{\bar{m}}$. Ξ is a subset of $\mathfrak{R}^{\bar{m}}$, F is a probability measure generated by a distribution function also denoted by F and $\mathcal{F}$ is the completion for F of the Borel algebra in $\mathfrak{R}^{\bar{m}}$. We shall assume that Ξ is convex. If this was not the case, we then replace it by its convex hull which we will also denote Ξ and fill up $\mathcal{F}$ with the appropriate sets of measure zero. Without loss of generality, we can assume that Ξ is of full dimension. If not, we can change Problem (1) so as to include the deterministic second-stage constraints, into the set of fixed first-stage constraints. Then, our new Ξ has full dimension. The probability distribution function F is continuous, discrete or a mixture of both.

In view of the interpretation given to (1), it is easy to see that the second-stage decision (control) variable v is a function of the observed state of the system, viz. $d - T(u,\xi)$, and in particular a function of the random variable ξ. Thus, v is itself a random variable. This fact is expressed by our notation $v[\xi]$. Moreover, we do not make any assumptions on v as a function of ξ, e.g., as to its measurability. Since by the nature of the model it is "calculated" only for the value of ξ which actually occurs. We will, however, show that $E_\xi\{\mathrm{Inf}\ q(v[\xi])\}$ makes sense.

In what follows we show that there exists an equivalent problem to (1), i.e. a problem with the same set of optimal solutions as (1), that can be expressed as the minimization of a convex functional on a convex set.

B. <u>The Second Stage Problem</u>

Once u is selected and ξ is observed, the second stage problem

$$\text{(2)} \quad \text{Find} \quad \inf q(v) \quad \text{subject to} \quad W(v) = d - T(u,\xi), \quad v \in V$$

becomes a deterministic problem.

Let

$$V(u,\xi) = \{v \mid W(v) + T(u,\xi) = d,\ v \in V\} \tag{3}$$

be the set of feasible solutions for (2), and le:

(4) $$Q(u,\xi) = \operatorname{Inf}\{q(v) \mid v \in V(u,\xi)\}$$

be the functional describing the range of the infimum of $q(v)$ as a function of u and ξ. As we shall see later, we may restrict ourselves to the case when $V(u,\xi)$ is non-empty. The set $V(u,\xi)$ is convex and closed, but not necessarily compact. Thus, the functional $q(v)$ may fail to achieve its minimum on $V(u,\xi)$. We shall assume that $q(v)$ possesses finite infimum on $V(u,\xi)$. Such a condition is not very restrictive, because if for some u, $Q(u,\xi) \equiv -\infty$ for all ξ in Ξ, then $z(u) = -\infty$ and Problem (1) is of no interest. Moreover, if for some u, $Q(u,\xi)$ equals minus infinity for a proper subset of Ξ, we could still hope that this set would have measure zero, and our problem could have a meaningful solution. But it is not the case, since we shall show that if $Q(u,\xi) = -\infty$ for some ξ in Ξ, then $Q(u,\xi) \equiv -\infty$ for all ξ in Ξ. To do so, we need the following results:

(5) PROPOSITION: Fix u and let $V(u,\xi) \neq 0$ for all ξ in Ξ, then $Q(u,\xi)$ is a convex function in ξ on Ξ.

Proof: For $\varepsilon > 0$, we say that v determines an ε- inf of $q(v)$ on $V(u,\xi)$ if $v \in V(u,\xi)$ and $q(v) \leq Q(u,\xi) + \varepsilon$.

First, we shall assume that $Q(u,\xi) > -\infty$ for all ξ in Ξ. Let $\xi_0, \xi_1 \in \Xi$, then $\lambda\xi_0 + (1 - \lambda)\xi_1 = \xi_\lambda \in \Xi$ for $\lambda \in [0,1]$. Let v_0 and v_1 determine ε - inf on $V(u,\xi_0)$ and $V(u,\xi_1)$, respectively. By the convexity of V and linearity of the operators W and T,

$\lambda v_0 + (1 - \lambda)v_1 \in V(u,\xi_\lambda)$ for $\lambda \in [0,1]$.

Then

$$Q(u,\xi_\lambda) \leq q(\lambda v_0 + (1 - \lambda)v_1)$$

also, by the convexity of the functional q,

$$q(\lambda v_0 + (1 - \lambda)v_1) \leq \lambda q(v_0) + (1 - \lambda)q(v_1)$$

and since v_0 and v_1 determine ε - inf, we have

$$\lambda q(v_0) + (1 - \lambda)q(v_1) \leq \lambda Q(u,\xi_0) + (1 - \lambda)Q(u,\xi_1) + \varepsilon$$

i.e.

$$Q(u,\xi_\lambda) \leq \lambda Q(u,\xi_0) + (1 - \lambda)Q(u,\xi_1) + \varepsilon.$$

Since, the above inequality holds for any ε, arbitrarily close to zero, we obtain

$$Q(u,\xi_\lambda) \leq \lambda Q(u,\xi_0) + (1 - \lambda)Q(u,\xi_1).$$

Let us now consider the case when $Q(u,\xi)$ is not finite for all ξ in Ξ. Without loss of generality, we can assume that $Q(u,\xi_0) = -\infty$. If $Q(u,\xi_0) = -\infty$, then for all N arbitrarily large, there exists $v_0 \in V(u,\xi_0)$ such that $q(v_0) \leq -N$. But

$$Q(u,\xi_\lambda) \leq q(\lambda v_0 + (1 - \lambda)v_1)$$

and by convexity of the functional q

$$q(\lambda v_0 + (1-\lambda)v_1) \le \lambda q(v_0) + (1-\lambda)q(v_1)$$

and since $Q(u,\xi_0) = -\infty$, $\exists v_0$ such that

$$\lambda q(v_0) + (1-\lambda)q(v_1) \le -N$$

for any N; thus

$$Q(u,\xi_\lambda) \le -N$$

i.e. $Q(u,\xi_\lambda) = -\infty$ for $\lambda \in (0,1)$.

This implies that if there exists some ξ in Ξ such that $Q(u,\xi)$ has no lower bound, then $Q(u,\xi) \equiv -\infty$ for every ξ in the interior on Ξ and $Q(u,\xi)$ may be different from $-\infty$ at most on the boundaries of Ξ.

(6) PROPOSITION: If for a fixed u, $V(u,\xi) \neq \emptyset$ for all ξ in Ξ and at least one of the three following assumptions is satisfied:

(i) $q(v)$ is linear and V is a convex polyhedral subset of $\mathfrak{R}^n$,

(ii) V is compact

(iii) $q(v)$ is weakly continuous on V and V is weakly compact

(iv) Ξ is open

then $Q(u,\xi)$ is continuous in ξ on Ξ.

<u>Proof</u>: Since $Q(u,\xi)$ is convex, $Q(u,\xi)$ is continuous on the interior of Ξ (this proves the proposition under assumption (iv)). Thus, the only case of interest is when ξ is on the boundary of Ξ. The proposition under Assumption (i) is proved in [8]. We limit ourselves to (ii) and (iii).

Let $\xi_0 \in \delta\Xi$ and $\xi_i \to \xi_0$, where each ξ_i belongs to the interior of Ξ. Under either (ii) or (iii) there exists a subsequence v^{i_k} such that $q(v^{i_k}) \to q(v^0)$ for some v^0 in V and such that $W(v^0) + T(u,\xi_0) = d$, where v^i is an ε-inf corresponding to $_i$. Hence, $\lim\limits_{i \to \infty} Q(u,\xi_i) \geq \{\mathrm{Inf}\ q(v) \mid v \in V(u, \lim\limits_{i \to \infty} \xi_i)\} = Q(u,\xi_0)$. On the other hand, by the convexity of $Q(u,\xi)$, we have

$$Q(u,\xi_0) \geq \lim_{k \to \infty} Q(u,\xi_k)$$

thus

$$\lim_{k \to \infty} Q(u,\xi_k) = Q(u,\xi^0).$$

<u>Remark</u>: The conditions (i), (ii), (iii) or (iv) are sufficient conditions to ensure the continuity of $Q(u,\xi)$. They are not necessary. In general, however, $Q(u,\xi)$ may fail to be continuous in ξ, as is shown by the following example, where V is of finite dimension. Let $V = \mathfrak{R}^2_+$, $\Xi = [0,1]$, $q(v) = q(x,y) = -\mathrm{Min}\ (|\sqrt{x\ y}\ |,1)$, $d = 0$

$$W(v) = x \quad \text{and} \quad T(u,\xi) = -\xi.$$

It is easy to see that $Q(u,\xi) = -1$ if $\xi \neq 0$ and $Q(u,0) = 0$. Hence, $Q(u,\xi)$ is not continuous for $\xi = 0$.

(7) <u>Corollary</u>: For a fixed u, let $V(u,\xi) \neq \emptyset$ for all ξ in Ξ. If $Q(u,\xi) = -\infty$ for some ξ in Ξ and at least one of the conditions (i), (ii), (iii) or (iv) of Proposition (6) is satisfied, then $Q(u,\xi) \equiv -\infty$ for all ξ in Ξ.

In what follows, we shall assume that either Ξ is open--or it can be redefined so that it is open--or that at least one of the conditions (i), (ii), or (iii) of Proposition (6) holds.

C. <u>The Solution Set</u>

A fixed u and an observed ξ determines $Q(u,\xi)$ uniquely, then our only decision variable is u. It is in this context that we examine the solution set of Problem (1). Nonetheless, the second-stage decisions affect our first-stage decision, not only by the values assumed by $Q(u,\xi)$, but also by the restriction that we have to limit our set of admissible first-stage decision to those for which there exist a feasible second-stage decision.

(8) <u>Definition</u>: u is a *feasible solution to (1)*, if $A(u) = b$, $u \in U$, and if the feasibility of Problem (2) is independent of the value assumed by ξ in Ξ. Let K be the set of feasible solutions to (1). Let $K_1 = \{u \in \mathcal{U} \mid A(u) = b\} \cap U$, be the set determined by the *fixed constraints*.

(9) PROPOSITION: K_1 is a closed convex subset of $\mathcal{U}$.

<u>Proof</u>: By linearity and continuity of the operator A and convexity of the closed set U.

Let $K_2 = \{u \in U \mid \xi \in \Xi, V(u,\xi) \neq \emptyset\}$ be the set representing the *induced constraints*. By induced, we mean that the set K_2 is determined by a condition to be satisfied at some later time, viz.: the second-stage problem must be feasible for all ξ in Ξ.

Let $K_{2\xi} = \{u \in U | V(u,\xi) \neq \emptyset\}$, then $K_2 = \bigcap_{\xi \in \Xi} K_{2\xi}$. By the linearity of the operators W and T and convexity of V, $K_{2\xi}$ is convex. Thus,

(10) PROPOSITION: K_2 is a convex subset of U.

Note that introducing the appropriate sets of measure zero, in order to replace the original Ξ by its convex hull, does not change the set K_2. Let $\tilde{\Xi}$ be the original probability space and let Ξ be its convex hull. Let $\tilde{K}_2 = \bigcap_{\xi \in \tilde{\Xi}} K_{2\xi}$ and K_2 as above. Obviously, $\tilde{K}_2 \supset K_2$ since the intersection is taken over a smaller index set. Thus, it suffices to show that $u \in \tilde{K}_2$ implies that $u \in K_2$. If $u \in \tilde{K}_2$, then $u \in K_{2\xi}$ for all ξ in $\tilde{\Xi}$, and $V(u,\xi) \neq \emptyset$ for all ξ in $\tilde{\Xi}$. If $\hat{\xi} \in \Xi$, but not to $\tilde{\Xi}$, then there exists ξ_1, ξ_2 in $\tilde{\Xi}$ such that $\hat{\xi} = \lambda\xi + (1 - \lambda)\xi_2$ for some $\lambda \in (0,1)$. By the linearity of W and T and since $V(u,\xi_1)$ and $V(u,\xi_2)$ are non-empty so is $V(u,\xi)$. Thus, $V(u,\xi) \neq \emptyset$ for all ξ in Ξ, i.e. $u \in K_2$.

(11) PROPOSITION: The set K of feasible solution to Problem (1), is a convex subset of U.

Proof: $K = K_1 \cap K_2$.

D. The Objective Function

To show that (1) can be reduced do an equivalent convex program, it now suffices to show that $z(u)$ --the objective function of Problem (1)-- is a convex function in u on K. Remark that $u \in K$ implies that $V(u,\xi)$ is non-empty for all ξ in Ξ.

(12) PROPOSITION: $Q(u,\xi)$ is convex in u on K.

Proof: Fix ξ and take $u_0, u_1 \in K$, then $\lambda u_0 + (1-\lambda)u_1 = u_\lambda \in K$. Since we assumed that $Q(u,\xi) > -\infty$, there exists v_0 and v_1 which determine ε- inf of $q(v)$ on $V(u_0,\xi)$ and $V(u_1,\xi)$, respectively. Also, by convexity of V and linearity of W and T, $\lambda v_0 + (1-\lambda)v_1 \in V(u_\lambda,\xi)$. Then

$$Q(u_\lambda,\xi) \leq q(\lambda v_0 + (1-\lambda)v_1) \leq \lambda q(v_0) + (1-\lambda)q(v_1) \leq \lambda Q(u_0,\xi) + (1-\lambda)Q(u_1,\xi) + \varepsilon.$$

Since this inequality holds for all ε, we have

$$Q(u_\lambda,\xi) \leq \lambda Q(u_0,\xi) + (1-\lambda)Q(u_1,\xi) \quad \text{for } \lambda \in [0,1].$$

(13) PROPOSITION: Let $Q(u) = E_\xi\{Q(u,\xi)\}$. Then $Q(u)$ is convex in u on K.

Proof: The function $Q(u,\xi)$ is continuous, thus Lebesque measurable. But F is a Lebesque-Stieltze measure and $\mathcal{G}$ contains the Borel Algebra, thus $Q(u,\xi)$ is also F-measurable. Since F determines a positive measure, $Q(u)$ is the result of a weighted positive linear combination of convex functions. Thus $Q(u)$ is convex.

Since $c(u)$ is convex, we have shown that there exists an equivalent convex program to Problem (1), viz.

$$\boxed{\begin{array}{ll} \text{Find} \quad \inf \quad z(u) = c(u) + Q(u) \\ \text{subject to} \qquad\qquad u \in K \end{array}} \tag{14}$$

where no random elements are present any longer. Nonetheless, two main difficulties remain to be solved before one could use efficiently the techniques available for convex programs, namely: Depending on the structure of the different operators of the original problem, to find an explicit expression for $Q(u)$ and the set K, may be a major undertaking. As we shall show in a forthcoming paper, a certain and interesting class of problems allow us to express $Q(u)$ and K explicitly, with relatively little effort.

3. DUALITY

A. The Dual Problem

Solution methods for any particular problem of the form (1) depend strongly on the form of the operators involved. However, as was the case in linear programming under uncertainty [8], there is a dua ·ty theory which plays a crucial role.

The second-stage problem, (once u is selected and ξ is observed),

$$\begin{aligned} &\text{Find} \quad \inf \; q(v) \\ &\text{subject to} \quad W(v) = d - T(u,\xi) \\ &\qquad\qquad\quad v \in V \end{aligned}$$

and the equivalent convex program.

$$\begin{aligned} &\text{Find} \quad \inf \; c(u) + Q(u) \\ &\text{subject to} \quad A(u) = b \\ &\qquad\qquad\quad u \in K_2 \cap U \end{aligned}$$

are in the same form. To develop the duality theory for this class of problems, it suffices to consider the following simple problem.

$$\begin{aligned} &\text{Find} \quad \inf \; c(u) \\ &\text{subject to} \quad A(u) = b \\ &\qquad\qquad\quad u \in U \subset \mathcal{U}, \end{aligned} \tag{15}$$

where b, $c(u)$, U and $\mathcal{U}$ are as defined in the previous section. We remember in particular that U is closed and convex, and A is a continuous linear operator with range in $\mathfrak{R}^m$.

Let

$$C = \{p = (p_0,p_1,\ldots,p_m) \mid p_0 \geq c(u),\ (p_1,\ldots,p_m) = A(u) - b,\quad u \in U\}.$$

and

$$\mathfrak{C} = \{(p_1,\ldots,p_m) \mid (p_1,\ldots,p_m) = A(u) - b,\quad u \in U\}$$

(16) <u>Lemma</u>: C is convex.

<u>Proof</u>: Suppose $p^1,p^2 \in C$ and suppose further that $u^1,\ u^2 \in U$ satisfy

$$p_0^i \geq c(u^i),\qquad (p_1^i,\ldots,p_m^i) = A(u^i) - b,\qquad i = 1,2.$$

Let $p^\lambda = \lambda p^1 + (1 - \lambda)p^2$ and $u^\lambda = u^1 + (1 - \lambda)u^2$ for $\lambda \in [0,1]$.

Then

$$\lambda(p_1^1,\ldots,p_m^1) + (1 - \lambda)(p_1^2,\ldots,p_m^2) = \lambda(A(u^1) - b) + (1 - \lambda)(A(u^2) - b)$$

$$= A(u^\lambda) - b.$$

Also, by convexity of the functional c we have

$$c(u^\lambda) \leq \lambda c(u^1) + (1 - \lambda)c(u^2) \leq \lambda p^1 + (1 - \lambda)p^2$$

Unfortunately, it is not true, in general, that C is closed.

Consider (15) with

$$\mathcal{U} = \ell_2 = \{\{u_i\} \mid \Sigma u_i^2 < \infty\}$$

$$A(u) = u_1$$

$$c(u) = \sum_{i=2}^{\infty} \frac{1}{2^i} u_i^2$$

$$U = \{u \mid \sum_{i=2}^{\infty} u_i^2 = 1\}$$

For any b in $\mathfrak{C}$, $\inf c(u) = 0$, but there exists no feasible u such that $c(u) = 0$. In particular, let u^i, $i=2,\ldots$ be given by $u^i_j = \delta_{ij}$. Then $A(u^i) = 0$ and $c(u^i) = \frac{1}{2^i}$, $i=2\ldots$. Thus, we have $p^i = (p^i_0, p^i_1) \geq (\frac{1}{2^i}, 0) \to (0,0) = p^0$ where $p^0 \notin C$ and each p^i does. However, we will need $\mathfrak{C}$ closed. This will be the case when U is weakly sequentially compact or if it is a convex polyhedron subset of a finite Euclidian space. In general, we see that we are essentially seeking to find the "lowest" point of C, on the p_0 axis. That is, we can reformulate (15) as follows:

$$\text{(17)} \qquad \begin{array}{ll} \text{Find} & \inf p_0 \\ \text{subject to} & p \in L \cap C \end{array}$$

where $L = \{(p_0, p_1, \ldots, p_m) \mid p_i = 0,\ i=1,\ldots,m\}$

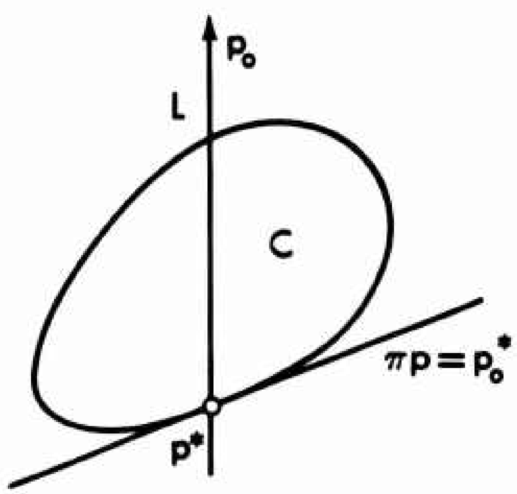

Figure 18

Problem (17) has the very natural dual.

(19) Find $\sup \mu$

subject to $\pi_0 = 1$

$\pi p - \mu \geq 0$ for all p in C,

where μ is a scalar, and π_0 is the first component of the $m + 1$ dimensional vector π.

If we think of $\pi p - \mu = 0$ as defining a hyperplane in $\mathfrak{R}^m$, then there is a one-to-one correspondence between feasible solutions of (19) and non-vertical supporting hyperplanes which are "below" the set C, in the sense that increasing p_0 means up.

Immediately, we have

(20) <u>Proposition (Weak Duality)</u>: $p_0 \geq \mu$ for all feasible solutions to (17) and (19).

<u>Proof</u>: $\pi p - \mu \geq 0$ by (19). Since p is feasible for (17), then $p_1 = \cdots = p_m = 0$ and hence $\pi_0 p_0 - \mu \geq 0$. But $\pi_0 = 1$.

We now prove the following intuitively obvious duality theorem:

(21) <u>Theorem (Strong Duality)</u>: If the projection $\mathfrak{C}$ of C with respect to p_0 is closed, exactly one of the following occurs:

a) (17) and (19) both admit feasible solutions in which case $\inf p_0 = \sup \mu$,

b) (17) is feasible and (19) is not in which case $\inf p_0 = -\infty$,

c) (19) is feasible and (17) is not in which case $\sup \mu = +\infty$,

d) neither (17) nor (19) is feasible.

<u>Proof</u>: (a) By (20) $\inf p_0$ and $\sup \mu$ are finite. Let $\mu^* = \inf\{p_0 \mid p \in L \cap C\} > -\infty$. Clearly there exists p^* which belongs to $\bar{C} \cap L$ such that $\mu^* = p_0^*$ and p^* is a boundary point of $\tilde{C} = \{(p_0,\dots,p_m) \mid p_i = p_i', i=1,\dots,m, p_0 \geq p_0', p' \in C\}$. Hence, there exists a supporting hyperplane to $\tilde{C}$ at p^*. Let it be defined by $\hat{\pi}p - \hat{\mu} \geq 0$. Clearly $\hat{\pi}_0 \geq 0$. If $\hat{\pi}_0 > 0$ division by $\hat{\pi}_0$ yields $\pi p - \mu \geq 0$ where $\pi = \frac{1}{\hat{\pi}_0}(\hat{\pi})$, $\mu = \frac{\hat{\mu}}{\hat{\pi}_0}$ for all $p \in C$. Since $\pi p^* - \mu = 0$ implies $p_0^* = \mu^* = \mu$; (π,μ) is optimal for (19) and $\inf p_0 = \sup \mu$. If for every supporting hyperplane of $\tilde{C}$ at p^*, we have that $\hat{\pi}_0 = 0$ a somewhat more complicated construction is necessary. Let $\varepsilon > 0$ be arbitrary, and let $\tilde{p} = (p_0^* - \varepsilon, 0,\dots,0)$, then $\tilde{p} \notin \bar{C}$. Hence there exists a hyperplane separating strictly $\tilde{p}$ and $\bar{C}$, i.e. there exists $\tilde{\pi},\tilde{\mu}$ such that $\tilde{\pi}\tilde{p} - \tilde{\mu} < 0$ and $\tilde{\pi}p - \tilde{\mu} \geq 0$ for all p in $\bar{C}$. In particular, $\pi p^* - \mu \geq 0$ but $\tilde{\pi}\tilde{p} = \tilde{\pi}_0\tilde{p}_0 = \tilde{\pi}_0(p_0^* - \varepsilon) < \tilde{\pi}_0 p_0^*$ implies that $\tilde{\pi}_0 > 0$. Letting $\pi = \frac{1}{\tilde{\pi}_0}\tilde{\pi}$ and $\mu = \tilde{p}_0 = p_0^* - \varepsilon$, we have a feasible solution to (19) with $\mu = p_0^* - \varepsilon$. Since ε is arbitrary $\sup \mu = \inf p_0 = p_0^*$.

(b) If $\inf p_0 > -\infty$, a feasible solution to (19) exists by the same construction as in a), but by hypothesis no feasible solution to (19) exists, therefore $\inf p_0 = -\infty$.

(c) Suppose $\sup \mu < +\infty$, then let $\tilde{\mu} = \sup\{\mu \mid \pi_0 = 1, \pi p - \mu \geq 0, \forall p \in C\}$ and let $\tilde{p} = (\tilde{\mu},0,\dots,0)$. We now establish that $\tilde{p} \notin \tilde{C}$. In fact, $\tilde{p} \notin \bar{C}$, for if it did $(\tilde{p}_1,\dots,\tilde{p}_m) = (0,\dots,0)$ would belong to $\bar{\mathfrak{C}} = \mathfrak{C}$. But then

$(c(0),0,\dots,0)$ is a feasible point for (17) which is assumed infeasible. Hence, $\tilde{p} \notin \tilde{C}$. Therefore, there is a hyperplane separating strictly $\tilde{p}$ and $\tilde{C}$ determined by, say $\tilde{\pi},\tilde{\mu}$, and such that

$$\tilde{\pi}\tilde{p} < \inf\{\tilde{\pi}p \mid p \in \tilde{C}\} \leq \inf\{\tilde{\pi}p \mid p \in C\}.$$

By definition of $\tilde{C}$ and since C is non-empty, we have the $\tilde{\pi}_0 \geq 0$. If $\tilde{\pi}_0 > 0$, let π,μ be given by

$$\pi = \frac{1}{\tilde{\pi}_0} \cdot \tilde{\pi} \text{ and } \frac{\tilde{\pi}\tilde{p}}{\pi_0} = \tilde{\mu} < \mu \leq \frac{1}{\tilde{\pi}_0} \inf\{\tilde{\pi}p \mid p \in C\}$$

π,μ is a feasible solution to (19) with $\mu > \tilde{\mu}$ which contradict the definition of $\tilde{\mu}$. Suppose now that $\tilde{\pi}_0 = 0$. Then $\tilde{\pi}\tilde{p} = 0$, hence $\tilde{\pi}p > \delta > 0$ for all $p \in \tilde{C}$. Let $\bar{\pi},\bar{\mu}$ be any feasible solution to (19), then $\pi = (\bar{\pi} + \lambda\tilde{\pi})$, $\mu = (\bar{\mu} + \lambda\delta)$ is also feasible for any λ. Taking any $\lambda > 0$ contradicts the definition of $\tilde{\mu}$.

(22) <u>Corollary</u>: If p^* and π^*,μ^* are respectively feasible for (17) and (19) they are optimal iff

$$\pi^*p^* = \mu^*.$$

<u>Proof</u>: $(\pi^*p^*) = \langle (1,\pi_1^*,\dots,\pi_m^*), (p_0^*,0,\dots,0) \rangle = p_0^*$, hence $p_0^* = \mu^*$ but $p_0 \geq \mu$ for all feasible solutions of (17) and (19) by Proposition (20). In particular, $p_0^* \geq \sup \mu$. Since $\mu^* = p_0^*$, (π^*,μ^*) is optimal. Conversely, $\inf\ p_0 \geq \mu^*$ and since $p_0^* = \pi^*,p^*$ is optimal. On the other hand, if p^* and π^*,μ^* are optimal, i.e. they achieve the inf and sup in (17) and (19) then by Theorem (21) they must satisfy $\pi^*p^* = \mu^*$.

(23) Corollary (Pre-maximum principle) If p^* is optimal for (17), there exists a π^* such that:

$$\pi^* p^* = \mathrm{Min}\{\pi^* p \mid p \in C\}.$$

Proof: Clearly p^* is a boundary point of C then $\exists$ supporting hyperplane $\pi^* p - \mu \geq 0$ for all $p \in C$ with $\pi^* p^* - \mu = 0$.

Remark: If in Corollary (23) $\pi_0^* > 0$, then $\frac{\pi^*}{\pi_0^*}$, p_0^* determines an optimal solution to (17); however, this need not be the case (see Figure 24).

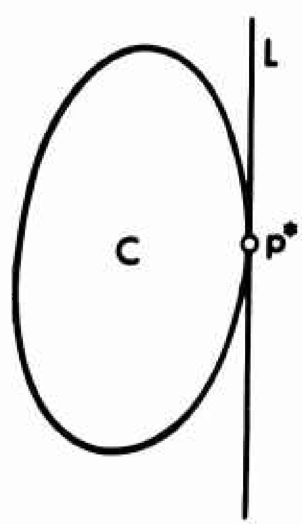

Figure 24

(25) <u>Corollary</u>: If C is closed and the infimum in (17) exists and is finite, then the infimum is attained for a feasible solution.

(26) <u>Corollary</u>: If L intersects the relative interior of C and (19) is feasible, the supremum in (19) is attained.

<u>Proof</u>: [6].

B. <u>Special Cases</u>

We now apply the results of the last section to the original problem (15) and examine some special cases. We first interpret the dual problem (19). For the special case (18), it is equivalent to

(27)

$$\begin{aligned} &\text{Find} \quad \sup \mu \\ &\text{s.t.} \quad c(u) - \pi[A(u) - b] \geq \mu \quad \text{for all} \quad u \in U. \end{aligned}$$

An easy lemma is:

(28) <u>Lemma</u>: If $c(u) = c \cdot u$ is linear and U is a cone, then for any feasible π, μ for (27) we have

$$[c - \pi A](u) \geq 0 \quad \text{for all} \quad u \in U.$$

(29) <u>Application to linear programs</u>: Consider the linear program

$$\begin{aligned} &\text{Min} \quad cx \\ &\text{s.t.} \ Ax = b \\ &\qquad x \geq 0. \end{aligned}$$

By direct application of (27), its "dual" is

(30) Find $\sup \mu$

$$\text{s.t.} \quad cx - \pi[Ax - b] \geq \mu \text{ for all } x \geq 0.$$

Since the set of non-negative x is a cone (28) applies and we have

$[c - \pi A]x \geq 0$ for all $x \geq 0$. Further, by taking $x = I_i$ $i=1,\ldots,m$, where I_i is the i^{th} unit vector. We obtain

$$c - \pi A \geq 0 \text{ for any feasible } \pi.$$

Re-arranging (30) we obtain

$$(c - \pi A)x + \pi b \geq \mu.$$

Clearly for a given π, the largest μ is given by

$$\mu - \pi b = \operatorname*{Inf}_{x \geq 0} (c - \pi A)x = 0.$$

Hence, the dual problem becomes

(31) $\sup \pi b$

$$\text{s.t.} \quad c - \pi A \geq 0.$$

To obtain the usual duality theorem for linear programming from (21) it suffices to observe that if $\sup \pi b < \infty$, then it is attained by some feasible π, and similarly for the primal objective.

(32) Application to linear control problems: Consider a dynamical system, the evolution of which, is described by the ordinary linear differential equations

$$\frac{dx(t)}{dt} = A(t)x(t) + u(t) \text{ on the time interval}$$

$[0,T]$ where $x(t) = (x_0(t),\ldots,x_n(t))$, $A(t)$ is a $n+1 \times n+1$ matrix of continuous functions and $u(t) = (u_0(t),\ldots,u_n(t))$ is a vector of controls. For simplicity, we assume that $u \in U = \{u \mid u(t) \in \Omega$ $0 \leq t \leq T$, and u is measurable and bounded$\}$ where Ω is a closed convex subset of $\mathfrak{R}^{n+1}$. Further we assume that

$$x_i(0) = x_i^0 \qquad i=0,\ldots,n$$

and

$$x_i(T) = x_i^T \qquad i=1,\ldots,n.$$

The value of $x_0(T)$ is not prescribed and the problem is to minimize $x_0(T)$ over all $x(t)$ and $u(t)$ satisfying the above relations.

As is well known, $x(T) = Y(T)x^\circ + \int_0^T Y(T)Y^{-1}(s)\, u(s)ds$ where $Y(T)$ is a $n+1 \times n+1$ matrix of functions satisfying the adjoint equation:

$$Y(t) = -Y(t)A(t),\ Y(0) = I.$$

It is easily seen that $S_T = \{x(T) \mid x(T) = Y(T)x(0) + \int_0^T Y(T)Y^{-1}(s)u(s)ds,\ u \in U\}$

is convex. Thus, the duality theory previously developed can be used.

If $x^*(t)$, $u^*(t)$, is an optimal solution to the control problem, we can apply Corollary (23) yielding the existence of $\pi = (\pi_0,\ldots,\pi_{m+1})$ such that

$$\pi x^*(t) = \mathrm{Min}\{\pi x \mid x \in S_T\}.$$

Using the particular form of the description of the set S_T, we have:

$$\pi\left[Y(t)x^{\circ} + \int_0^T Y(T)Y^{-1}(s)u^*(s)ds\right]$$

$$\leq \pi Y(T)x^{\circ} + \int_0^T \pi Y(T)Y^{-1}(s)u(s)ds$$

for all $u \in \Omega$. Thus

$$0 \leq \int \pi Y(T)Y^{-1}(s)[s(s) - u^*(s)]ds.$$

If we define $\Pi(t) = \pi Y(T)Y^{-1}(s)$, it is easily seen [6] that

$u^*(s) = \underset{u \in \Omega}{\text{Min}}\ \Pi(s)u$ and $\Pi(t)$ is a vector solution of the adjoint equation, which is equivalent to the maximum principle for this problem.

REFERENCES

[1] A. Balakrishnan: "Optimal Control Problems in Banach Spaces". To appear in SIAM Journal on Control, Vol. 3, 1965.

[2] G. B. Dantzig: "Linear Programming under Uncertainty", Management Sciences, Vol. 1, 197-206, 1955.

[3] G. B. Dantzig and A. Madansky: "On the Solution of Two-Stage Linear Programs under Uncertainty", Proceedings of the Fourth Symposium on Mathematical Statistics and Probability, Vol. 1, University of California, Berkeley, 1961.

[4] H. Kushner: "On the Stochastic Maximum Principle: Fixed Time of Control", RIAS Report, 1963.

[5] A. Madansky: "Dual Variables in Two-Stage Linear Programming under Uncertainty", J. Math. Analysis and Applications, Vol. 6, 98-108, 1963.

[6] R. Van Slyke: "Mathematical Programming and Optimal Control", Ph. D thesis, University of California, Berkeley, 1965.

[7] R. Wets: "Programming under Uncertainty: The Complete Problem", Boeing document D1-82-0379, Seattle, Washington, 1964. To appear in Zeitschrift für Warscheinlichkeitstheorie und Verwandte Gebiete.

[8] R. Wets: "Programming under Uncertainty: The Eqivalent Convex Problem", Boeing document D1-82-0411, Seattle, Washington, 1965.

[9] A. Williams: "A Stochastic Transportation Problem", J. Operations Research, Vol. 11, 759-770, 1963.

[10] A. Williams: "On Stochastic Linear Programming". To appear in SIAM, 1965.